First-passage percolation, non-positive curvature, and radial maps

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BACKGROUND:
A way to perturb the metric on a graph $X = (V, E)$ is to assign random i.i.d. lengths to its edges, a process called first-passage percolation (FPP). These lengths are classically interpreted as the passage times of the corresponding edges. A typical question in this area asks whether the perturbed graph, X_ω, admits bi-infinite geodesics. For FPP on hyperbolic graphs, the authors in [BT17], answer this question in the positive. For FPP on $\mathbb{Z}^d$, the answer is expected to be negative [WW98]. We ask whether other geometric phenomena arising from non-positive curvature are preserved.

DEFINITION:
$f : X \to Y$ is a **radial map** with respect to $o \in X \iff$ $\exists \lambda_+, \mu_+, \lambda_-, \mu_- > 0$ such that
- $d_Y(f(x), f(y)) \le \lambda_+ d_X(x, y) + \mu_+, \forall x, y \in X,$
- $d_Y(f(x), f(y)) \ge \lambda_- d_X(x, y) - \mu_-, \forall x, y \in \mathrm{Im}(\gamma),$ $\forall$ finite geodesics $\gamma: [0, M] \to X, \gamma(0) = o$ [DV15].

SETUP:
Let $X = (V, E)$ be an infinite connected graph of bounded degree. Fix a distribution ν on $[0, \infty)$. To each $e \in E$, assign independently at random a variable $\omega(e)$ according to ν. Let $x, y \in V$ and γ be any edge path joining x and y. Define the ω-length of γ as $\ell_\omega(\gamma) := \sum_{e \in \gamma} \omega(e)$. The induced (psuedo)metric on X_ω is then $d_\omega(x, y) = \inf_{\gamma} \ell_\omega(\gamma)$. Assume $\nu(\{0\}) = 0$ and $\mathbb{E}\omega(e) < \infty$; we show the following:

(ABRIDGED) THEOREMS:
(1) Suppose X is hyperbolic; a.s. X_ω is not hyperbolic.
(2) Suppose X is coarsely $CAT(0)$; a.s. X_ω is not coarsely $CAT(0)$.
(3) Suppose α is a Morse ray; a.s. $\omega(\alpha)$ admits no Morse gauge.
(4) Let $o \in V$. For a.e. ω, the canonical map $(X, d) \to (X_\omega, d_\omega)$ is a radial map with respect to o.

DISCUSSION:
Although first-passage percolation preserves the combinatorial structure of a graph, it generally destroys its non-positive curvature. Nevertheless, along geodesic rays emanating from a fixed basepoint, FPP behaves like a quasi-isometry. In [BM22], the authors prove that for hyperbolic groups, the FPP "velocity" exists in almost every direction of the Gromov boundary. We ask whether an analogous result holds for FPP on the relatively hyperbolic groups considered in [GGT20].

First-passage percolation (FPP) destroys aspects of **non-positive curvature** (but is a **radial map**).

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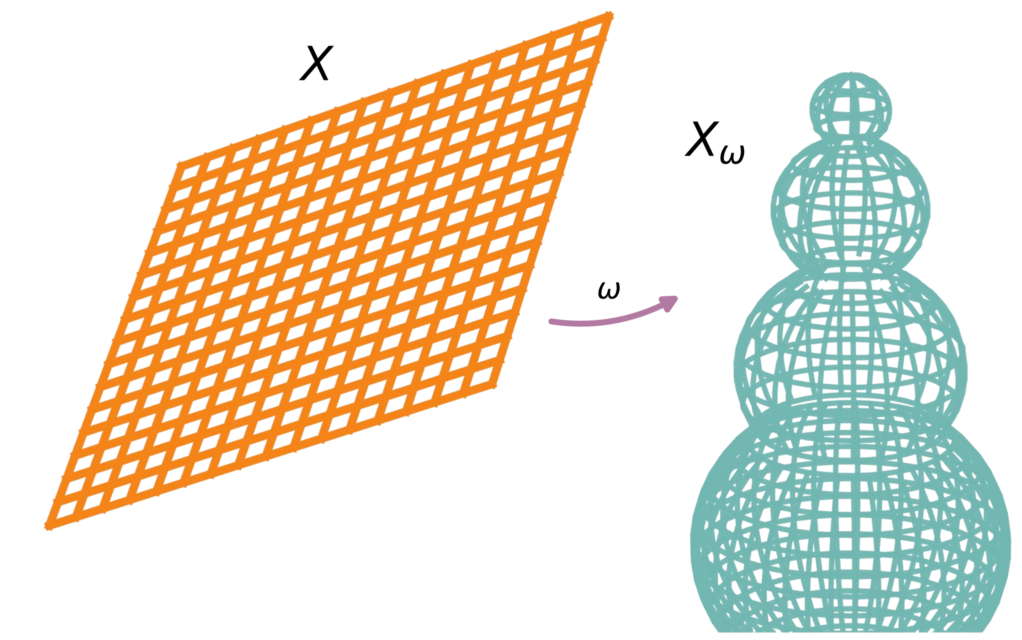

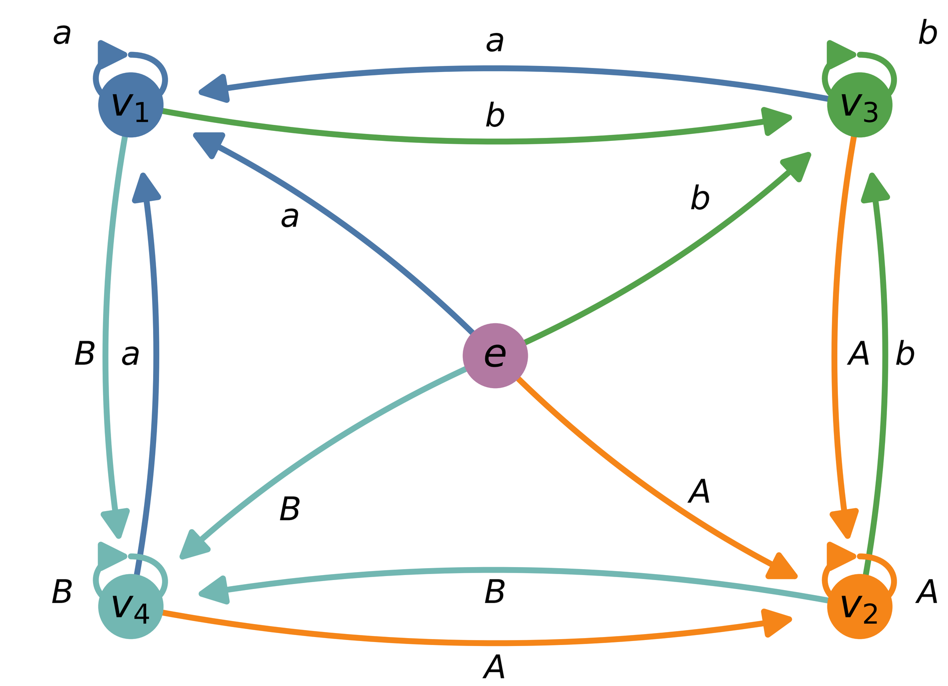

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